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Kelvin–Voigt vs Fractional Derivative Model as Constitutive Relations for Viscoelastic Materials

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Three simple constitutive relationships are studied for application to viscoelastic materials. Experimental results for both a rubbery and a glassy viscoelastic material are fit by the three schemes. The Kelvin–Voigt scheme is shown to be adequate in only limited frequency ranges. A three-parameter fractional order constitutive relationship provides a substantially better model over a much larger bandwidth. A four-parameter fractional model improves on the accuracy in materials with significant glassy regions.

Introduction

VISCOELASTIC materials are of interest in a wide variety of applications from passive damping to aircraft tire construction. Good modeling of a material's behavior is essential to the study and accurate design incorporating the particular material. Previously, viscoelastic materials have been characterized primarily with the Kelvin–Voigt model[1,2] for the constitutive relationship. Rogers[3] used a complex rational polynomial interpolation scheme. Torvik and Bagley[4] and Bagley and Torvik[5,6] have developed models using fractional derivatives. Padovan[7] and others have examined various issues involved in numerical implementation of these sorts of models.

These models are compared within this paper for a variety of real materials. The fractional derivative model is shown to be substantially more accurate and appropriate for modeling materials of interest in tire design. This improved accuracy is true for both narrow and broad frequency ranges. Accuracy over a wide frequency range is extremely desirable for aircraft tires because of temperature effects and high-vibration mode numbers.

Viscoelastic Materials

Viscoelastic behavior occurs in a wide range of materials that show some sort of liquid-like elastic behavior. Such materials include acrylics, rubber, and glass. A conventional Hooke's law linear elastic constitutive relationship is not an accurate representation of viscoelastic material behaviors such as creep and stress relaxation over short time scales. Further, the material behavior is dependent on its own time history. Viscoelastic materials are commonly said to have "memory" because of this phenomena.

A Hooke's law style constitutive relationship transformed into the frequency domain is given by

$$\bar{\sigma} = E^*\bar{\varepsilon} = (E_1 + iE_2)\bar{\varepsilon} \tag{1}$$

where E^* is the complex modulus, $\bar{\sigma}$ stress, and $\bar{\varepsilon}$ strain; E_1 and E_2 are the real and imaginary parts, respectively, of the complex modulus. It is common to use two combinations of these components when testing a viscoelastic material,[1,2] the modulus M and the loss factor L.

$$M = |E^*| = \sqrt{E_1^2 + E_2^2} \tag{2}$$

$$L = E_2/E_1 \tag{3}$$

Elsewhere in this paper appropriate subscripts will be used with these quantities to indicate which constitutive model the equation represents. The loss factor is a measurement of the rate of energy dissipation by the material.

It should be emphasized that the complex modulus E^* can vary signficantly with frequency. It is this frequency dependence which is difficult to model. As a side note, E^* also changes with change in temperature. These temperature effects can be interpreted as an appropriate shift in frequency.[2,3] For aircraft tire applications both effects are very significant. The critical mode for an aircraft tire often has a high-mode number which becomes critical at high speed. Aircraft tires also undergo very significant temperature changes in normal use. Typical aircraft operations range in broad terms up to 260°C (500°F) (Ref. 8) and 200 Hz. These two points make accurate modeling of the complex modulus over wide frequency ranges critical for tire applications.

Kelvin–Voigt Model

The Kelvin–Voigt constitutive relationship models a material as a lumped parameter system similar to a spring and dashpot in parallel. This constitutive relationship leads to

$$\sigma = q_0\varepsilon + q_1\dot{\varepsilon} \tag{4}$$

where σ is material stress, ε is strain, and q_0 and q_1 are material parameters that are to be determined. This differential equation can be restated, using a Fourier transform, as

$$\bar{\sigma} = (q_0 + i\omega q_1)\bar{\varepsilon} \tag{5}$$

From this it follows that $E_1 = q_0$, $E_2 = \omega q_1$, and

$$M_{\mathrm{KV}} = \sqrt{q_0^2 + \omega^2 q_1^2} \tag{6}$$

$$L_{\mathrm{KV}} = \omega(q_1/q_0) \tag{7}$$

The analysis reveals one of the critical failures of the Kelvin–Voigt scheme: a loss factor that is linear in frequency. Experimental testing, discussed later, shows that this is definitely not the case. Further, the Kelvin–Voigt scheme does not retain any time history or memory features.

Fractional-Order Models

A simple three-parameter fractional-order model is obtained by mimicking the Kelvin–Voigt model. The first derivative in time in the Kelvin–Voigt model is replaced by a fractional-order derivative of order α thus producing

$$\sigma = q_0\varepsilon + q_1 D^\alpha \varepsilon \tag{8}$$

This fractional-order constitutive model uses a fractional-order derivative defined as[9]

$$D^\alpha[x(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{x(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau, \qquad 0 < \alpha < 1 \tag{9}$$

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This integro-differential operator has "fading" or decaying memory due to the $(t-\tau)^{-\alpha}$ term. It is also reasonably easy to work with when transformed,

$$\mathcal{F}[D^\alpha(x)] = (i\omega)^\alpha \mathcal{F}(x) \tag{10}$$

Transforming Eq. (8) yields,

$$\bar{\sigma} = \left[q_0 + q_1(i\omega)^\alpha\right]\bar{\varepsilon} \tag{11}$$

Now the modulus and loss factor become

$$M_3 = \sqrt{q_0^2 + q_1^2\omega^{2\alpha} + 2q_0q_1\omega^\alpha\cos(\alpha\pi/2)} \tag{12}$$

$$L_3 = \frac{q_1\omega^\alpha\sin(\alpha\pi/2)}{q_0 + q_1\omega^\alpha\cos(\alpha\pi/2)} \tag{13}$$

Thus, the loss factor is a nonlinear function of the frequency. As will be shown later in this paper, this approach models the experimental results much better than the Kelvin–Voigt.

An analysis of the asymptotic behavior of these terms at low frequency ($\omega \to 0$) shows that the modulus approaches q_0. The loss factor approaches zero like $(q_1/q_0)\omega^\alpha\sin(\alpha\pi/2)$. This kind of result is as expected. A static load is met by some stiffness q_0, but there is no energy loss. Note, the asymptotic behavior of the Kelvin–Voigt model near $\omega = 0$ is identical, albiet with a somewhat different value for q_0.

Similarly, at high frequency ($\omega \to \infty$) the modulus grows like $q_1\omega^\alpha$. Because $0 < \alpha < 1$ (typically α is between $\frac{1}{2}$ and $\frac{2}{3}$), this growth in modulus is much slower than the linear growth [Eq. (7)] obtained from the Kelvin–Voigt model. At high frequency the loss factor approaches the constant $\tan(\alpha\pi/2)$, unlike the linear growth seen in the Kelvin–Voigt model.

As the three-parameter model was unable to capture a decrease in loss factor which is seen in some materials in the high-frequency (or "glassy") region, the four-parameter fractional model

$$\sigma + p_1D^\alpha\sigma = q_0\varepsilon + q_1D^\alpha\varepsilon \tag{14}$$

was also examined. Transforming yields

$$\bar{\sigma} = \frac{q_0 + q_1(i\omega)^\alpha}{1 + p_1(i\omega)^\alpha}\bar{\varepsilon} \tag{15}$$

the corresponding modulus and loss factor are found to be

$$M_4 = \frac{\sqrt{\left[q_0 + (q_1 + q_0p_1)\omega^\alpha\cos(\alpha\pi/2) + p_1q_1\omega^{2\alpha}\right]^2 + \left[(q_1 - q_0p_1)\omega^\alpha\sin(\alpha\pi/2)\right]^2}}{1 + 2p_1\omega^\alpha\cos(\alpha\pi/2) + p_1^2\omega^{2\alpha}} \tag{16}$$

$$L_4 = \frac{(q_1 - q_0p_1)\omega^\alpha\sin(\alpha\pi/2)}{q_0 + (q_1 + q_0p_1)\omega^\alpha\cos(\alpha\pi/2) + p_1q_1\omega^{2\alpha}} \tag{17}$$

At the low-frequency limit ($\omega \to 0$), this model exhibits the same behavior as the three-parameter model with the modulus approaching q_0 and the loss factor vanishing like $(q_1/q_0 - p_1)\omega^\alpha\sin(\alpha\pi/2)$. However, at the high-frequency limit the modulus now approaches the constant q_1/p_1 whereas the loss factor vanishes like $(1/p_1 - q_0/q_1)\omega^{-\alpha}\sin(\alpha\pi/2)$. This is the expected behavior for a glassy material which shows stiffness at very high frequency but no energy loss.

The asymptotic study of the models' behaviors is also useful for gaining physical understanding of the meaning of the various model parameters. It is clear, for instance from the constitutive relationship, that for the Kelvin–Voigt model q_0 is Young's modulus and q_1 is a damping parameter. For the fractional models, q_0 is obviously analogous to Young's modulus, particularly at low frequency. At higher frequencies, the other parameters contribute not only to the material damping but also to the stiffness.

The parameter α is the primary driver in the shape of the modulus and loss factor curves. As $\alpha \to 1$, the three-parameter fractional model's loss factor approaches being linear. Thus, as one would expect, as $\alpha \to 1$ the three-parameter model approaches the Kelvin–Voigt model. As α moves away from 1 the loss factor curve bends over and grows much slower with frequency.

Results

Experimental data[10] for two very different materials was used. The first is a material somewhat like tire rubber. As one would expect, it has "rubbery" behavior over a broad frequency range. The second material is an acrylic core foam. It is typically used for damping applications and, thus, shows a much more obvious glassy region than does the rubber.

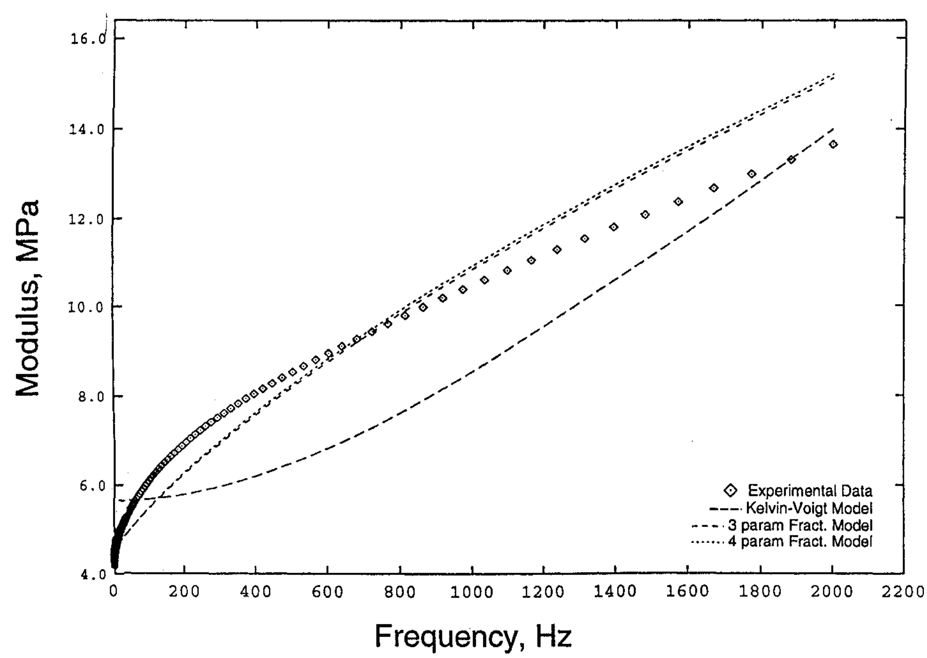

Fig. 1 Modulus variation with frequency for rubber material.

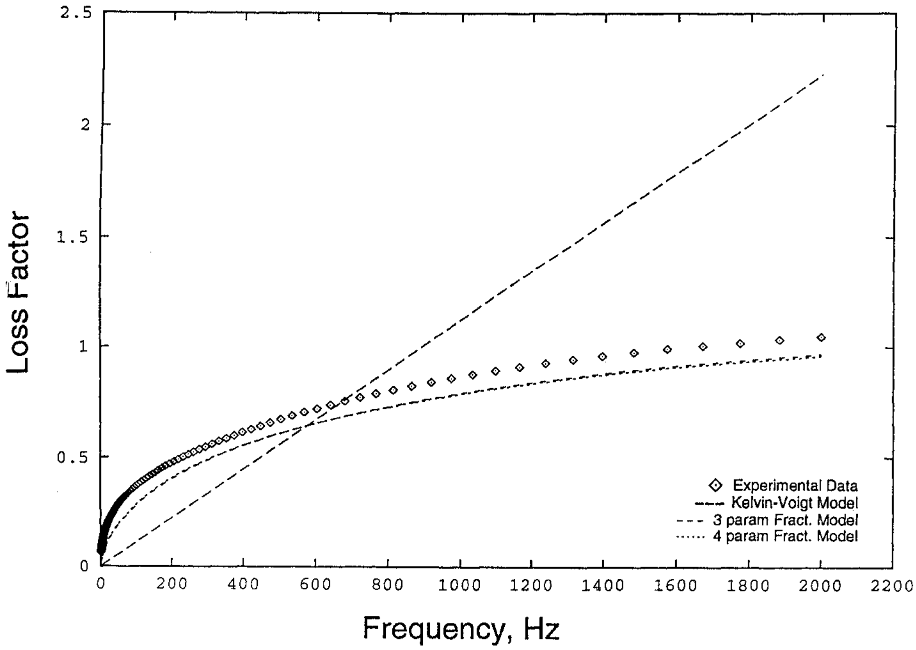

Fig. 2 Loss factor variation with frequency for rubber material.

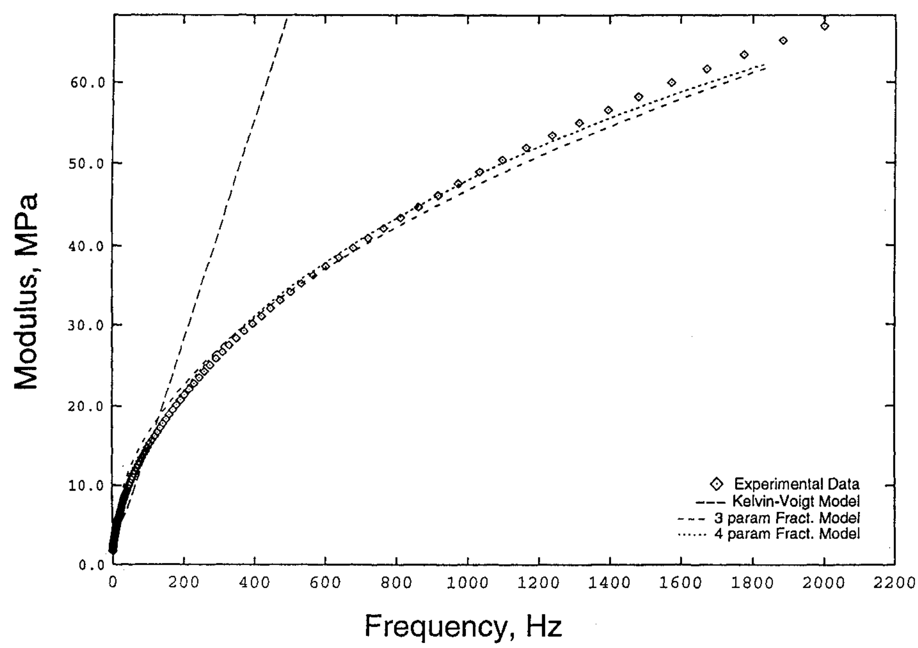

Fig. 3 Modulus variation with frequency for acrylic material.

Table 1 Rubber material constants

Parameter	Kelvin–Voigt	3 Parameter	4 Parameter
q_0	0.8355	0.6437	0.6437
q_1	9.632×10^{-4}	1.359×10^{-2}	1.359×10^{-2}
p_1	n/a	n/a	0.0
α	n/a	0.6442	0.6442

Table 2 Acrylic material constants

Parameter	Kelvin–Voigt	3 Parameter	4 Parameter
q_0	1.8887	0.0	0.1189
q_1	4.117×10^{-3}	0.2946	0.1581
p_1	n/a	n/a	6.350×10^{-3}
α	n/a	0.4558	0.5800

A least squares fit is performed to calculate the material parameters (q_0, q_1, p_1, and α as appropriate) for each of the three models. This fit is performed to minimize the error between the model and the experimental complex modulus E^*. Note, the Kelvin–Voigt style analog to the four-parameter fractional model was within the permissible space for the least squares optimizer (i.e., α was allowed to equal one if needed). However, the least squares fit chose α to be well away from one. Tables 1 and 2 give each of the material parameters as fit by the computer code.

A couple things should be noted from these tables. First, the data used for rubber did not included a significant glassy region. This means that the added flexibility of the four-parameterfractional model was not needed, and the value of p_1 was optimized to be zero. Thus, the two fractional models are the same for this material.

For the acrylic foam, the three-parameter fractional model was unable to accurately represent the glassy high-frequency region. As the asymptotic discussion showed, the three parameter model loss factor monotonically increases to a constant. Thus, the three-parameter model is inappropriate for this material. The zero value for q_0 is an indication of the optimizing code struggling to fit the data as well as possible and a warning of a bad choice of model.

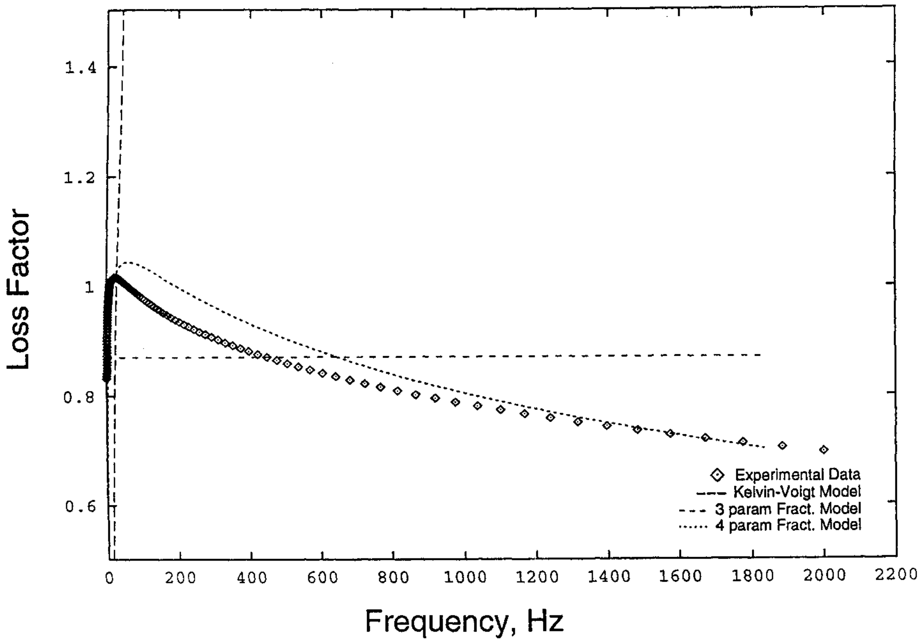

Fig. 4 Loss factor variation with frequency for acrylic material.

As q_0 is the asymptotic value for E^* as $\omega \rightarrow 0$, a value of $q_0 = 0$ indicates zero static stiffness which is not physically accurate.

Figure 1 shows the variation of the modulus for rubber with frequency. It can be seen that the two fractional models do a substantially better job of matching the experimental data[9] for the material behavior over the entire range than does the Kelvin–Voigt model. The fractional models show accurately shaped curves and physically similar asymptotic behavior to the experimental results. Recall that the three-parameter and four-parameter results are identical for this material.

Although its modeling of the physical behavior of the material over the entire frequency is poor, the Kelvin–Voigt model's fit for the modulus is better than the fractional models' fits at very high frequency. As will be seen in Fig. 2, the Kelvin–Voigt model's fit for the loss factor in this region is extremely poor. The important issue is how well a particular scheme models the behavior of the complex modulus E^*, whose components can be studied by plotting the modulus and loss factor.

Figure 2 illustrates the variation of the loss factor for rubber with frequency. It is immediately obvious that the Kelvin–Voigt's linear approximation to the loss factor is completely inappropriate and inaccurate. The fractional model does a much better job of capturing its variation over frequency.

The acrylic foam's modulus behavior is shown in Fig. 3. Again, the two fractional models show substantially better performance than does the Kelvin–Voigt model. Note that despite the nonphysical result of the three-parameter model having zero stiffness at zero frequency, the modulus fit for this model is very good. The next figure illustrates the true problem with the model for this material.

The loss factor for the acrylic is illustrated in Fig. 4. Again, the Kelvin–Voigt scheme does an extremely poor job of modeling the material's loss factor. The three-parameter fractional model does a good job of capturing the material's rubbery region behavior, but is unable to fit the decrease in loss factor at high frequency. The four-parameter model is able to capture this glassy region much better.

Conclusions

Accurate modeling of viscoelastic materials is important in a wide variety of applications. Cost and ease of use are also important considerations.

The Kelvin–Voigt model has been widely used due to its simplicity and ease of implementation. Unfortunately, its fidelity over a broad frequency variation is extremely poor. Its prediction of a linear variation in loss factor with frequency is not physically accurate and could result in severely flawed designs. As mentioned previously, the loss factor is a measure of the energy dissipated by the material. Errors in its prediction lead to errors in modeling the damping and energy loss (i.e., temperature) of a system.

In general, the fractional models do a substantially better job of capturing the real behavior of the materials over a wide frequency range. They also contain the "fading memory" feature that viscoelastic materials exhibits. Unfortunately, this feature is also a problem.

The time integral in the fractional derivative definition specifies integration from time zero to the current time. A brute force implementation of a fractional model would require storage of the entire time history of the solution. This could quickly become prohibitively expensive. We are currently considering various schemes to significantly reduce this numerical overhead.

Acknowledgment

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